Definite integral. Fundamental theorem of Calculus

November 21, 2016

Problems

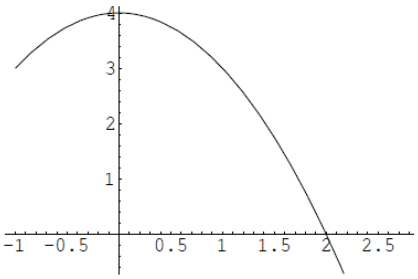
Problem 1. Compute the integral $\int_1^2 x^k dx$ **by definition** using the following partition.

For $r := 2^{1/n}$, define the partition to be $1 < r < r^2 < \cdots < r^{n-1} < r^n = 2$. The purpose of this problem is to show that a particular choice of the partition to work with can make a huge difference. Try to compute this integral by using the standard partition and see for yourself how complicated it will be.

Problem 2. True or False:

1. If $\int f(x)dx = \int g(x)dx$ for continuous functions f and g, then $f(x) = g(x)$.
2. If f and g are differentiable and $f'(x) = g'(x)$, then $f(x) = g(x)$.
3. $\frac{d}{dx}\left(\int_a^b f(x)dx\right) = f(x)$.

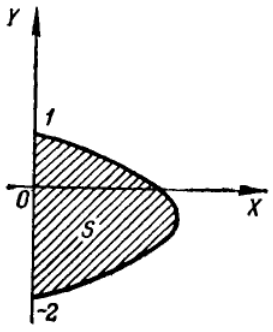
Problem 3. Below if the graph of a function f.



Let $g(x) = \int_0^x f(t)dt$ then for $0 < x < 2$ the function $g(x)$ is

1. increasing and concave up;
2. increasing and concave down;
3. decreasing and concave up;
4. decreasing and concave down.

Problem 4. Evaluate the area bounded by the curve $x = 2 - y - y^2$ and the y-axis. (Hint: try switching the roles of the coordinates.)



Problem 5. Compute the area bounded by the curve $y = \frac{1}{x}$, the x-axis, the straight line $x = e$ and the horizontal line $y = 2$. (Hint: draw the picture.)

Problem 6. Using definite integrals, find the limit of the following sum:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$$

(Hint: $\dfrac{1}{n+i} = \dfrac{1}{n}\cdot\dfrac{1}{1+\frac{i}{n}}$)

Problem 7. Determine the signs of the following integrals without evaluating them:

1. $\displaystyle\int_{-1}^{3} x^2 dx$
2. $\displaystyle\int_{0}^{2\pi} \tfrac{\sin x}{x} dx$

Problem 8. Determine (without evaluating) which of the following integrals is greater:

1. $\displaystyle\int_0^1 \sqrt{1+x^2}dx$ or $\displaystyle\int_0^1 dx$
2. $\displaystyle\int_0^1 x^2\sin^2 x dx$ or $\displaystyle\int_0^1 x\sin^2 x dx$

Problem 9. Let f be a function on $[0,1]$ defined as follows:

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ is a rational number, } \mathrm{g.c.d.}(p,q) = 1 \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

If this function integrable? (Hint: try to pick specific partitions similar to the example of Dirichlet function.)